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Wess-Zumino model with exact supersymmetry on the lattice

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Abstract: A lattice formulation of the four dimensional Wess-Zumino model that uses Ginsparg-Wilson fermions and keeps exact supersymmetry is presented. The supersymmetry transformation that leaves invariant the action at finite lattice spacing is determined by performing an iterative procedure in the coupling constant. The closure of the algebra, generated by this transformation is also showed.

Keywords: Lattice Quantum Field Theory, Supersymmetry Breaking.

Contents

1. Introduction

Non-perturbative studies of supersymmetric theories turn out to have remarkably rich properties which are of great physical interest. For this reason, much effort has been dedicated to formulating a lattice version of supersymmetric theories. See for example [1]–[9] and [10, 11] for recent reviews. While much is known analytically, the hope is that the lattice would provide further information and confirm the existing analytical calculations.

The major obstacle in formulating a supersymmetric theory on the lattice arises from the fact that the supersymmetry algebra is actually an extension of the Poincaré algebra, which is explicitly broken by the lattice. Indeed, in an interacting theory, translation invariance is broken since the Leibniz rule is not valid for lattice derivatives [1]. Ordinary Poincaré algebra is also broken by the lattice but the hypercubic crystal symmetry forbids relevant operators which could spoil the Poincaré symmetry in the continuum limit. In the case of the super Poincaré algebra, the lattice crystal group is not enough to guarantee the absence of supersymmetry violating operators. Without exact lattice supersymmetry one might hope to construct non-supersymmetric lattice theories with a supersymmetric continuum limit. This is the case of the Wilson fermion approach for the $N = 1$ supersymmetric Yang-Mills theory where the only operator which violates the $N = 1$ supersymmetry is a fermion mass term. By tunning the fermion mass to the supersymmetric limit one recovers supersymmetry in the continuum limit [9, 12]. Alternatively, using domain wall fermions [13] or overlap fermions [14], this fine tunning is not required. Recently, a new lattice construction for models with extended supersymmetry has been proposed [15]. In this case, the lattice preserves some supersymmetries which are enough to reduce or eliminate the need for fine tunning (see also [16]).

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In the past the lattice Wess-Zumino model has been perturbatively studied using Wilson fermions [3, 4] and adding to the action a Wilson term also for the scalar fields. In the continuum limit this results in a cancellation of divergences between fermion and scalar fields. However, scalar and fermion renormalization wave functions in general do not coincide, due to finite contributions, thus in order to restore supersymmetry in the continuum limit a fine tuning of the various coupling of the lattice action is needed [3]. For the two dimensional case this problem is not present [4], at least in perturbation theory, where the continuum supersymmetric Ward identities are recovered in the limit of vanishing lattice spacing without a fine tuning.[1]

More recently, a lattice Wess-Zumino model has been defined in refs. [6, 7] using a general Ginsparg-Wilson operator. In this case the supersymmetric continuum limit is recovered without a fine-tuning also in four dimensions [7]. Moreover, this formulation allows to consider Yukawa interactions which are invariant under lattice chiral transformation [18], thus it appears to be suitable for chiral theories and, in particular, for supersymmetric gauge theories.

In this paper we consider the four dimensional lattice Wess-Zumino model introduced in refs. [6, 7] and show that it is actually possible to formulate the theory in such a way that the full action is invariant under a lattice superymmetry transformation at a fixed lattice spacing. The action and the transformation are written in terms of the Ginsparg-Wilson operator and reduce to their continuum expression in the naive continuum limit $a \to 0$. The lattice supersymmetry transformation is non-linear in the scalar fields and depends on the parameters m and g entering in the superpotential. We also show that the lattice supersymmetry transformation close the algebra, which is a necessary ingredient to guarantee the request of supersymmetry. We believe that the existence of this exact symmetry is responsible for the restoration of supersymmetry in the continuum limit, which has been explicitly verified in perturbation theory in the case of the scalar and fermion two-point functions [7].

The paper is organized as follows. In section 2 we introduce the Ginsparg-Wilson fermion operator and formulate the lattice Wess-Zumino action. In section 3 we show how to build up a lattice supersymmetry transformation that is an exact symmetry of this model. In section 4 the closure of the algebra, crucial step to be satisfied in order to impose supersymmetry, is shown. Discussions and outlook are summarized in section 5. In appendices A, B and C, some details of the calculations are presented.

2. The Wess-Zumino model

The Ginsparg-Wilson relation [19]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \tag{2.1}$$

implies a continuum symmetry of the fermion action which may be regarded as a lattice form of the chiral symmetry [18]. As a matter of fact, the fermion lagrangian with a Yukawa

[1]Non-perturbative effects may produce supersymmetry breaking at finite volume [17].

interaction

$$\mathcal{L} = \bar{\psi} D \psi + g \bar{\psi} \left(P_+ \phi \hat{P}_+ + P_- \phi^\dagger \hat{P}_- \right) \psi \,, \tag{2.2}$$

where

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5) \,, \qquad \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5) \tag{2.3}$$

are the lattice chiral projection operators and $\hat{\gamma}_5 = \gamma_5(1 - aD)$, is invariant under the lattice chiral transformation

$$\delta\psi = i\varepsilon\hat{\gamma}_5\psi \,, \qquad \delta\bar{\psi} = i\bar{\psi}\gamma_5\varepsilon \,, \qquad \delta\phi = -2i\varepsilon\phi \,. \tag{2.4}$$

By writing ψ in terms of two Majorana fermions

$$\psi = \chi + i\eta \,, \tag{2.5}$$

it can be seen that the interaction term in eq. (2.2) couples the two Majorana fermions and therefore there is a conflict between lattice chiral symmetry and the Majorana condition [5, 6]. This is due to the fact that the projection operators $\hat{P}_\pm$ depend on D. Moreover, it has been observed that by making the following field redefinition

$$\psi' = \left(1 - \frac{a}{2} D\right) \psi \,, \qquad \bar{\psi}' = \bar{\psi} \,, \tag{2.6}$$

the Yukawa interaction becomes

$$g\bar{\psi}'(P_+\phi P_+ + P_-\phi^\dagger P_-)\psi' \tag{2.7}$$

and the two Majorana components of ψ' decouple. Taking advantage of this property, one can define the four dimensional Wess-Zumino on the lattice with Majorana fermions [6].

We start with a lagrangian defined in terms of the Ginsparg-Wilson fermions on the $d = 4$ euclidean lattice. Our analysis is valid for all operators which satisfy eq. (2.1), however, in the following we will use the particularly simple solution given by [20]

$$D = \frac{1}{a}\left(1 - \frac{X}{\sqrt{X^\dagger X}}\right), \qquad X = 1 - aD_w \,, \tag{2.8}$$

where

$$D_w = \frac{1}{2}\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - \frac{a}{2}\nabla^\star_\mu\nabla_\mu \tag{2.9}$$

and

$$\begin{aligned} \nabla_\mu\phi(x) &= \frac{1}{a}(\phi(x + a\hat{\mu}) - \phi(x)) \\ \nabla^\star_\mu\phi(x) &= \frac{1}{a}(\phi(x) - \phi(x - a\hat{\mu})) \end{aligned} \tag{2.10}$$

are the forward and backward lattice derivatives, respectively.

Substituting eq. (2.9) in eq. (2.8) we find convenient to isolate in D the part containing the gamma matrices and write

$$D = D_1 + D_2 \tag{2.11}$$

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where

$$D_1 = \frac{1}{a}\left(1 - \frac{1 + \frac{a^2}{2}\nabla^\star_\mu \nabla_\mu}{\sqrt{X^\dagger X}}\right), \qquad D_2 = \frac{1}{2}\gamma_\mu \frac{\nabla^\star_\mu + \nabla_\mu}{\sqrt{X^\dagger X}} \equiv \gamma_\mu D_{2\mu}\,. \tag{2.12}$$

In terms of D_1 and D_2 the Ginsparg-Wilson relation (2.1) becomes

$$D_1^2 - D_2^2 = \frac{2}{a} D_1\,. \tag{2.13}$$

The action of the 4-dimensional Wess-Zumino model on the lattice has been introduced in refs. [6, 7] and can be re-written using the above notation as

$$\begin{aligned} S_{WZ} = \sum_x \Big\{ &\frac{1}{2}\bar{\chi}\left(1 - \frac{a}{2}D_1\right)^{-1} D_2\chi - \frac{2}{a}\phi^\dagger D_1 \phi + F^\dagger \left(1 - \frac{a}{2}D_1\right)^{-1} F + \frac{1}{2}m\bar{\chi}\chi + \\ &+ m(F\phi + (F\phi)^\dagger) + g\bar{\chi}(P_+\phi P_+ + P_-\phi^\dagger P_-)\chi + g(F\phi^2 + (F\phi^2)^\dagger)\Big\}, \end{aligned} \tag{2.14}$$

where ϕ and F are scalar fields and χ is a Majorana fermion which satisfies the Majorana condition

$$\bar{\chi} = \chi^T C \tag{2.15}$$

and C is the charge conjugation matrix which satisfies

$$C^T = -C\,, \qquad CC^\dagger = 1\,. \tag{2.16}$$

Moreover, our conventions are

$$\begin{aligned} C\gamma_\mu C^{-1} &= -(\gamma_\mu)^T \\ C\gamma_5 C^{-1} &= (\gamma_5)^T\,. \end{aligned} \tag{2.17}$$

It is easy to see that in the continuum limit, $a \to 0$, eq. (2.14) reduces to the continuum Wess-Zumino action

$$\begin{aligned} S = \int \Big\{ &\frac{1}{2}\bar{\chi}(\partial\!\!\!/ + m)\chi + \phi^\dagger \partial^2 \phi + F^\dagger F + m(F\phi + (F\phi)^\dagger) + \\ &+ g\bar{\chi}(P_+\phi P_+ + P_-\phi^\dagger P_-)\chi + g(F\phi^2 + (F\phi^2)^\dagger)\Big\}. \end{aligned} \tag{2.18}$$

3. The supersymmetric transformation

If one defines the real components by

$$\phi \to \frac{1}{\sqrt{2}}(A + iB)\,, \qquad F \to \frac{1}{\sqrt{2}}(F - iG) \tag{3.1}$$

the Wess-Zumino action (2.14) can be written as

$$S_{WZ} = S_0 + S_{\rm int}\,, \tag{3.2}$$

with

$$
\begin{aligned}
S_0 = \sum_x \Big\{ & \frac{1}{2}\bar{\chi}\left(1-\frac{a}{2}D_1\right)^{-1} D_2\chi - \frac{1}{a}(AD_1A + BD_1B) + \\
& + \frac{1}{2}F\left(1-\frac{a}{2}D_1\right)^{-1} F + \frac{1}{2}G\left(1-\frac{a}{2}D_1\right)^{-1} G \Big\}\,, \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
S_{\rm int} = \sum_x \Big\{ & \frac{1}{2}m\bar{\chi}\chi + m(FA+GB) + \frac{1}{\sqrt{2}}g\bar{\chi}(A+i\gamma_5 B)\chi + \\
& + \frac{1}{\sqrt{2}}g[F(A^2-B^2)+2G(AB)] \Big\}\,. \qquad (3.4)
\end{aligned}
$$

The free part of the action, S_0, is invariant under the lattice supersymmetry transformation

$$
\begin{aligned}
\delta A &= \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
\delta B &= -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
\delta\chi &= -D_2(A-i\gamma_5 B)\varepsilon - (F-i\gamma_5 G)\varepsilon \\
\delta F &= \bar{\varepsilon}D_2\chi \\
\delta G &= i\bar{\varepsilon}D_2\gamma_5\chi\,. \qquad (3.5)
\end{aligned}
$$

In fact, the variation of S_0 under the this transformation is

$$
\begin{aligned}
\delta S_0 = \sum_x \Big\{ & \bar{\chi}\left(1-\frac{a}{2}D_1\right)^{-1} D_2\left[-D_2(A-i\gamma_5 B)\varepsilon - (F-i\gamma_5 G)\varepsilon\right] - \frac{2}{a}\bar{\chi}\varepsilon D_1 A + \\
& + \frac{2i}{a}\bar{\chi}\gamma_5\varepsilon D_1 B + (\bar{\varepsilon}D_2\chi)\left(1-\frac{a}{2}D_1\right)^{-1} F + i(\bar{\varepsilon}D_2\gamma_5\chi)\left(1-\frac{a}{2}D_1\right)^{-1} G \Big\}\,.
\end{aligned}
$$

By using (2.17) and integrating by part,[2] this variation becomes

$$
\begin{aligned}
& \sum_x \Big\{ -\bar{\chi}\varepsilon\left[\left(1-\frac{a}{2}D_1\right)^{-1} D_2^2 + \frac{2}{a}D_1\right] A + i\bar{\chi}\gamma_5\varepsilon\left[\left(1-\frac{a}{2}D_1\right)^{-1} D_2^2 + \frac{2}{a}D_1\right] B - \\
& -\bar{\chi}\left(1-\frac{a}{2}D_1\right)^{-1} D_2(F-i\gamma_5 G)\varepsilon + \bar{\chi}D_2\varepsilon\left(1-\frac{a}{2}D_1\right)^{-1} F + i\bar{\chi}D_2\gamma_5\varepsilon\left(1-\frac{a}{2}D_1\right)^{-1} G \Big\} = 0\,,
\end{aligned}
$$

where we have used the Ginsparg-Wilson relation (2.13), which implies

$$
\left(1-\frac{a}{2}D_1\right)^{-1} D_2^2 = -\frac{2}{a}D_1\,. \qquad (3.6)
$$

As discussed in [7], the variation of $S_{\rm int}$ under (3.5) does not vanish because of the failure of the Leibniz rule at finite lattice spacing [1].

In order to discuss the symmetry properties of the lattice Wess-Zumino model one possibility is to modify the action by adding irrelevant terms which make invariant the full action. Alternatively, one can modify the supersymmetry transformation of eq. (3.5) in

[2]For instance, for any scalar function $\mathcal{F}$ one has $\mathcal{F}\bar{\varepsilon}D_2\chi = \bar{\chi}D_2\mathcal{F}\varepsilon$.

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such a way that the action (3.2) has an exact symmetry for a different from zero.[3] Since the transformation (3.5) leaves invariant the free part of the action, this modification must vanish for $g = 0$. Therefore, we introduce the following transformation

$$
\begin{aligned}
\delta A &= \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
\delta B &= -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
\delta \chi &= -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon + gR\varepsilon \\
\delta F &= \bar{\varepsilon}D_2\chi \\
\delta G &= i\bar{\varepsilon}D_2\gamma_5\chi
\end{aligned} \tag{3.7}
$$

where R is a function to be determined by requiring that the variation of the action vanishes. We make the assumption that R depends on the scalar and auxiliary fields and their derivatives and not on χ.

The variation of the Wess-Zumino action under the transformation (3.7) is

$$
\begin{aligned}
\delta S_{WZ} = \sum_x \Big\{ & g\bar{\chi}\left(1 - \frac{a}{2}D_1\right)^{-1} D_2 R\varepsilon - m\bar{\chi}\Big[D_2(A - i\gamma_5 B)\varepsilon + (F - i\gamma_5 G)\varepsilon - gR\varepsilon\Big] + \\
& + m(A\bar{\varepsilon}D_2\chi + F\bar{\chi}\varepsilon + iB\bar{\varepsilon}D_2\gamma_5\chi - iG\bar{\chi}\gamma_5\varepsilon) + \frac{g}{\sqrt{2}}\bar{\chi}(\bar{\varepsilon}\chi + \gamma_5(\bar{\varepsilon}\gamma_5\chi))\chi - \\
& - \sqrt{2}g\bar{\chi}(A + i\gamma_5 B)\Big[D_2(A - i\gamma_5 B)\varepsilon + (F - i\gamma_5 G)\varepsilon - gR\varepsilon\Big] \\
& + \frac{g}{\sqrt{2}}\Big[(A^2 - B^2)\bar{\varepsilon}D_2\chi + 2FA\bar{\chi}\varepsilon + 2iFB\bar{\chi}\gamma_5\varepsilon + \\
& \quad + 2iAB\bar{\varepsilon}D_2\gamma_5\chi + 2GB\bar{\chi}\varepsilon - 2iGA(\bar{\chi}\gamma_5\varepsilon)\Big]\Big\}.
\end{aligned}
$$

By using the Fierz identity, terms with four fermions cancel as in the continuum. Moreover, g independent terms cancel out after an integration by part, and one is left with

$$
\begin{aligned}
\delta S_{WZ} = \sum_x \Big\{ & g\bar{\chi}\left[\left(1 - \frac{a}{2}D_1\right)^{-1} D_2 R + mR\right]\varepsilon - \\
& - \frac{g}{\sqrt{2}}\Big[2\bar{\chi}(A + i\gamma_5 B)D_2(A - i\gamma_5 B)\varepsilon - \bar{\chi}D_2(A - i\gamma_5 B)^2\varepsilon\Big] + \\
& + \sqrt{2}g^2\bar{\chi}(A + i\gamma_5 B)R\varepsilon\Big\}.
\end{aligned} \tag{3.8}
$$

The function R is determined by imposing the vanishing of δS_{WZ}. By expanding R in powers of g

$$R = R^{(1)} + gR^{(2)} + \cdots \tag{3.9}$$

and imposing the symmetry condition order by order in perturbation theory, we find

$$R^{(1)} = \left(\left(1 - \frac{a}{2}D_1\right)^{-1} D_2 + m\right)^{-1} \Delta L \tag{3.10}$$

[3] A similar attempt has been proposed by Golterman and Petcher, [4], for the 2-dimensional Wess-Zumino model.

with

$$\Delta L \equiv \frac{1}{\sqrt{2}}(2(A+i\gamma_5 B)D_2(A-i\gamma_5 B) - D_2(A-i\gamma_5 B)^2)$$
$$= \frac{1}{\sqrt{2}}\Big\{2(AD_2A - BD_2B) - D_2(A^2-B^2)$$
$$+ 2i\gamma_5\Big[(AD_2B + BD_2A) - D_2(AB)\Big]\Big\}\,. \tag{3.11}$$

To order g^2 one has

$$R^{(2)} = -\sqrt{2}\left(\left(1-\frac{a}{2}D_1\right)^{-1} D_2 + m\right)^{-1} (A+i\gamma_5 B)\left(\left(1-\frac{a}{2}D_1\right)^{-1} D_2 + m\right)^{-1} \Delta L\,, \tag{3.12}$$

and for $n \geq 2$

$$R^{(n)} = -\sqrt{2}\left(\left(1-\frac{a}{2}D_1\right)^{-1} D_2 + m\right)^{-1} (A+i\gamma_5 B)R^{(n-1)}\,. \tag{3.13}$$

By inserting these results in eq. (3.9), the function R to be used in eq. (3.7) is therefore, the formal solution of

$$\left[\left(1-\frac{a}{2}D_1\right)^{-1} D_2 + m + \sqrt{2}g(A+i\gamma_5 B)\right] R = \Delta L\,. \tag{3.14}$$

Notice that, from the perturbative expressions (3.10) and (3.13) one realizes that $R \to 0$ for $a \to 0$, since ΔL vanishes in this limit. Indeed, ΔL is different from zero because of the breaking of the Leibniz rule for a finite lattice spacing.

4. The algebra

We now study the algebra associated to the lattice supersymmetry transformation (3.7) introduced in the previous section. In particular, carrying out the commutator of two supersymmetries we must find a transformation which is still a symmetry of the Wess-Zumino action, i.e. the transformations of the fields form a closed algebra, order by order in g. In this section, we explicitly check this fact up to order g^1, even though the calculation can be generalized to any order.

Two supersymmetry transformations on the scalar field A give

$$\delta_1\delta_2 A = \delta_1(\bar\varepsilon_2\chi)$$
$$= -\bar\varepsilon_2\left[D_2(A-i\gamma_5 B)\varepsilon_1 + (F-i\gamma_5 G)\varepsilon_1 - gR\varepsilon_1\right]$$

and their commutator yields

$$[\delta_2,\delta_1]A = -2\bar\varepsilon_1 D_2\varepsilon_2 A + g(\bar\varepsilon_1 R\varepsilon_2 - \bar\varepsilon_2 R\varepsilon_1)\,. \tag{4.1}$$

The order g^1 of the second term on the r.h.s. of (4.1) reads

$$g(\bar\varepsilon_1 R^{(1)}\varepsilon_2 - \bar\varepsilon_2 R^{(1)}\varepsilon_1) = \sqrt{2}g\bar\varepsilon_2\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\times$$
$$\times\Big[D_2(A^2-B^2) - 2(AD_2A - BD_2B)\Big]\varepsilon_1 \tag{4.2}$$

where we used (3.10). Then, the commutator of two supersymmetries on the scalar field A is

$$[\delta_2,\delta_1]A = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\Big\{D_{2\mu}A + \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\times \\ \times\Big[D_{2\mu}(A^2-B^2) - 2(AD_{2\mu}A - BD_{2\mu}B)\Big]\Big\}. \quad (4.3)$$

Similarly, the commutators of two supersymmetries on the other fields, up to terms of order g^1, are (see appendix A for some details)

$$[\delta_2,\delta_1]B = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\Big\{D_{2\mu}B + \sqrt{2}g\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\times \\ \times\Big[D_{2\mu}(AB) - (AD_{2\mu}B + BD_{2\mu}A)\Big]\Big\}, \quad (4.4)$$

$$[\delta_2,\delta_1]F = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\Big\{D_{2\mu}F - \frac{g}{\sqrt{2}}\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\times \\ \times\Big[D_{2\mu}(A^2-B^2) - 2(AD_{2\mu}A - BD_{2\mu}B)\Big]\Big\}, \quad (4.5)$$

$$[\delta_2,\delta_1]G = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\Big\{D_{2\mu}G - \sqrt{2}g\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\times \\ \times\Big[D_{2\mu}(AB) - (AD_{2\mu}B + BD_{2\mu}A)\Big]\Big\} \quad (4.6)$$

and

$$[\delta_2,\delta_1]\chi = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\Big\{D_{2\mu}\chi - \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)-D_2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\times \quad (4.7) \\ \times\left(D_2(A-i\gamma_5B)\gamma_\mu\chi + (A+i\gamma_5B)D_2\gamma_\mu\chi - D_2[(A-i\gamma_5B)\gamma_\mu\chi]\right)\Big\}.$$

Therefore, the general expression of these commutators is

$$[\delta_1,\delta_2]\Phi = \alpha^\mu P_\mu^\Phi(\Phi)\,, \qquad \Phi = (A,B,F,G,\chi)\,, \quad (4.8)$$

where $\alpha^\mu = -2\bar{\varepsilon}_2\gamma^\mu\varepsilon_2$ and $P_\mu^\Phi(\Phi)$ are polynomials in Φ defined as

$$P_\mu^\Phi(\Phi) = D_{2\mu}\Phi + O(g) \quad (4.9)$$

where the order g^1 contributions can be read in (4.3)–(4.7). We have verified that the closure works, i.e. the action (2.14) is invariant under the transformation

$$\Phi \to \Phi + \alpha^\mu P_\mu^\Phi(\Phi) \quad (4.10)$$

up to terms of order g^1. This calculation is sketched in appendix B. Notice that, in the

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continuum limit $D_{2\mu} \to \partial_\mu$ and the transformation (4.10) reduces to

$$\Phi \to \Phi + \alpha^\mu \partial_\mu \Phi \tag{4.11}$$

since terms in (4.3)–(4.7) of order g^1 vanish due to the restoration of the Leibniz rule as $a \to 0$.

Higher orders in g of the transformation (4.10) can be determined by using the expression for $R^{(n)}$ given in (3.13). The proof of the invariance of the Wess-Zumino action under the transformation (3.7) at any order in g can be similarly performed. Indeed, the closure of the algebra at any order in g should hold since the supersymmetry transformation (3.7) is an exact symmetry of the lattice Wess-Zumino action.

5. Conclusions

In this paper, we have presented a lattice formulation of the four dimensional Wess-Zumino model with an exact supersymmetry using Ginsparg-Wilson fermions. We have shown that it is actually possible to formulate the theory in such a way that the full action is invariant under a lattice supersymmetry transformation at a fixed lattice spacing. This supersymmetry transformation introduces a function R which is non-linear in the scalar fields and depends on the parameters m and g entering in the interaction part of the action. The action and the transformation, which have been written in terms of the Ginsparg-Wilson fermions, reduce to their continuum expression in the limit $a \to 0$. We have also shown that the lattice supersymmetry transformations close the algebra, as it is required by the supersymmetry. While the present work is confined to the proof to the order g^1, concerning the closure of the algebra, there are no obstructions to extending this procedure to higher orders in g.

The study of the Ward identities associated to the exact lattice supersymmetry we have introduced can be done by generalizing the analysis performed by Golterman and Petcher in [4] for the two dimensional Wess-Zumino model. In the appendix C, we have calculated a simple Ward identity up to order $O(g)$ and verified that it is satisfied. We believe that the lattice supersymmetry we have introduced automatically leads to a restoration of the continuum supersymmetry without additional fine tuning. Explicit results on the two point functions [7] lend support to this idea and we are currently investigating on this issue in more detail.

Obviously one the most important question is whether these ideas may be extended to supersymmetric gauge theories where we expect that the Ginsparg-Wilson relation will play an important role in the construction of a lattice supersymmetry. However, there is an important difference between gauge theories and the Wess-Zumino model. The free and the interaction terms of a lattice gauge action are both contained in the plaquette term and therefore it is not obvious how to perform the iterative construction of the lattice supersymmetry transformation in this case.

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A. Some details of the supersymmetry algebra

In this appendix we show some details concerning the calculation of the commutator $[\delta_2, \delta_1]\chi$. Applying two supersymmetry transformations on χ one has

$$\delta_1\delta_2\chi == -D_2[\bar\varepsilon_1\chi - \gamma_5(\bar\varepsilon\gamma_5\chi)]\varepsilon_2 - [(\bar\varepsilon_1 D_2\chi) + \gamma_5(\bar\varepsilon_1 D_2\gamma_5\chi)]\varepsilon_2 + g(\delta_1 R)\varepsilon_2\,, \qquad \text{(A.1)}$$

where $(\delta_1 R)$ is the supersymmetry variation of the function R. A similar expression is obtained for $\delta_2\delta_1\chi$ with $\varepsilon_2 \leftrightarrow \varepsilon_1$. Terms to the order g^0 can be treated as in the continuum case by using a Fierz rearrangement as well as the relations in eq. (2.17). For instance, taking the first term in eq. (A.1) and the corresponding one with $\varepsilon_2 \leftrightarrow \varepsilon_1$ one has

$$-\bar\varepsilon_1 D_{2\mu}\chi(\gamma_\mu\varepsilon_2)_\alpha + \bar\varepsilon_2 D_{2\mu}\chi(\gamma_\mu\varepsilon_1)_\alpha = \frac{1}{2}(\bar\varepsilon_1\gamma_\mu\varepsilon_2)(D_2\gamma_\mu\chi)_\alpha - \frac{1}{4}(\bar\varepsilon_1\gamma_{\mu\nu}\varepsilon_2)(D_2\gamma_{\mu\nu}\chi)_\alpha\,. \quad \text{(A.2)}$$

Using a similar rearrangement for the remaining terms order g^0, the commutator of two supersymmetry transformations on χ is

$$[\delta_2, \delta_1]\chi_\alpha = -2\bar\varepsilon_1\gamma_\mu\varepsilon_2 D_{2\mu}\chi_\alpha + O(g)\,. \qquad \text{(A.3)}$$

The contribution to this commutator to the order g^1 is

$$\begin{aligned}(\delta_1 R^{(1)}\varepsilon_2 - \delta_2 R^{(1)}\varepsilon_1) &= \frac{\delta R^{(1)}}{\delta A}\varepsilon_2(\bar\varepsilon_1\chi) + \frac{\delta R^{(1)}}{\delta B}\varepsilon_2(\bar\varepsilon_1\gamma_5\chi) - (\varepsilon_1 \leftrightarrow \varepsilon_2)\\ &= \sum_R(\bar\varepsilon_1\gamma_R\varepsilon_2)\left(\frac{\delta R^{(1)}}{\delta A}\gamma_R\chi + \frac{\delta R^{(1)}}{\delta B}\gamma_R\gamma_5\chi\right) - (\varepsilon_1 \leftrightarrow \varepsilon_2) \qquad \text{(A.4)}\end{aligned}$$

where the sum is over the 16 independent 4×4 matrices. By using eq. (2.17) only the terms with $\gamma_R = \{\gamma_\mu, \gamma_{\mu\nu}\}$ survive, moreover, using the explicit form for $R^{(1)}$ one finds eq. (4.7).

B. A consistency check of the supersymmetry algebra

The prove of the invariance of the action under the transformation (4.9) to the order g^0 is immediate. In this appendix, we explicitly calculate the variation of the fermionic part of the action (3.2) to the order g^1 (the remaining part, containing only scalar fields, can be similarly treated).

$$\begin{aligned}&\delta\sum_x\left\{\frac{1}{2}\bar\chi(1-\frac{a}{2}D_1)^{-1}D_2\chi + \frac{1}{2}m\bar\chi\chi + \frac{1}{\sqrt2}g\bar\chi(A+i\gamma_5 B)\chi\right\} =\\ &\quad= -\frac{g}{\sqrt2}\sum_x\left\{\bar\chi\left((1-\frac{a}{2}D_1)^{-1}D_2 + m\right)\frac{m(1-\frac{a}{2}D_1) - D_2}{m^2(1-\frac{a}{2}D_1) + \frac{2}{a}D_1}\times\right.\\ &\qquad\qquad\times\Big(D_2(A-i\gamma_5 B)\gamma_\mu\chi + (A+i\gamma_5 B)D_2\gamma_\mu\chi - D_2[(A-i\gamma_5 B)\gamma_\mu\chi]\Big) -\\ &\qquad\qquad\left. - 2\bar\chi(A+i\gamma_5 B)D_{2\mu}\chi - \bar\chi(D_{2\mu}A + i\gamma_5 D_{2\mu}B)\chi\right\}, \qquad \text{(B.1)}\end{aligned}$$

where (4.3), (4.4) and (4.7) have been used. Due to the Ginsparg-Wilson relation (2.13), we have that

$$\left(\left(1-\frac{a}{2}D_1\right)^{-1}D_2+m\right)\frac{m(1-\frac{a}{2}D_1)-D_2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}=1 \tag{B.2}$$

and (B.1) becomes

$$\begin{aligned}-\frac{g}{\sqrt{2}}\sum_x\Big\{&\bar\chi D_2(A-i\gamma_5B)\gamma_\mu\chi+\bar\chi(A+i\gamma_5B)D_2\gamma_\mu\chi-\bar\chi\gamma_\mu(A-i\gamma_5\chi)D_2\chi\\&-2\bar\chi(A+i\gamma_5B)D_{2\mu}\chi-\bar\chi(D_{2\mu}A+i\gamma_5D_{2\mu}B)\chi\Big\}\end{aligned} \tag{B.3}$$

where in the last term of the first line an integration by part and the relation (2.17) has been used. Finally, terms containing the derivative of the scalar fields cancel out, while to prove the cancellation of the remaining ones a Fierz rearrangement is needed.

C. A simple Ward identity

By making a variation of the generating functional with respect to the lattice supersymmetry transformation (3.7), the Ward identity reads

$$\langle J_\Phi\delta\Phi\rangle=0 \tag{C.1}$$

where J_Φ are the sources for the fields $\Phi=(A,B,F,G,\chi)$. A simple Ward identity is

$$\langle D_2(A-i\gamma_5B)\rangle+\langle F\rangle-i\gamma_5\langle G\rangle-g\langle R\rangle=0\,. \tag{C.2}$$

The first term of this Ward identity is zero because of $\delta-$momentum conservation and $D_2(k=0)=0$. In the following, we show that this Ward identity is satisfied to order $O(g)$, i.e.

$$\langle F\rangle^{(1)}-i\gamma_5\langle G\rangle^{(1)}-g\langle R\rangle^{(0)}=0 \tag{C.3}$$

where $\langle\mathcal{O}\rangle^{(n)}$ denotes the expectation value of the function $\mathcal{O}$ to the n order in perturbation theory. From the action (3.2), the free propagators are

$$\begin{aligned}\langle AA\rangle^{(0)}&=\langle BB\rangle^{(0)}=-\mathcal{M}^{-1}\left(1-\frac{a}{2}D_1\right)^{-1}\\\langle FF\rangle^{(0)}&=\langle GG\rangle^{(0)}=\frac{2}{a}\mathcal{M}^{-1}D_1\\\langle AF\rangle^{(0)}&=\langle BG\rangle^{(0)}=m\,\mathcal{M}^{-1}\\\langle\chi\bar\chi\rangle^{(0)}&=-\mathcal{M}^{-1}\left(\left(1-\frac{a}{2}D_1\right)^{-1}D_2-m\right).\end{aligned} \tag{C.4}$$

The term $\langle F_x\rangle^{(1)}$ in momentum space reads (a factor $g/\sqrt{2}$ is omitted)

$$\begin{aligned}\langle F(k)\rangle^{(1)}=\int_{pq}\Big\{&\langle\bar\chi(p)\chi(q)\rangle^{(0)}\langle F(k)A(-p-q)\rangle^{(0)}+\\&+\langle F(k)F(-p-q)\rangle^{(0)}\left(\langle A(p)A(q)\rangle^{(0)}-\langle B(p)B(q)\rangle^{(0)}\right)+\\&+2\langle F(k)A(-p-q)\rangle^{(0)}\langle F(p)A(q)\rangle^{(0)}+\\&+2\langle F(k)A(-p-q)\rangle^{(0)}\langle B(p)G(q)\rangle^{(0)}\Big\}.\end{aligned} \tag{C.5}$$

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By substituting the propagators we have

$$\begin{aligned}\langle F(k)\rangle^{(1)} &= \frac{\delta^4(k)}{m}\int_p \mathcal{M}^{-1}(p)\,\mathrm{Tr}\left(\left(1-\frac{a}{2}D_1(p)\right)^{-1} D_2(p)-m\right)+\\ &\quad +4\delta^4(k)\int_p \mathcal{M}^{-1}(p)=0\,. \end{aligned}\tag{C.6}$$

The one point function of G is zero at this order.

Finally, the last term of the Ward identity (C.2) is

$$\langle R\rangle^{(0)} = \left(\left(1-\frac{a}{2}D_1\right)^{-1} D_2+m\right)^{-1}\langle \Delta L\rangle^{(0)}\,. \tag{C.7}$$

In the following we consider only the contribution from the field A since B can be treated similarly. In momentum space we have

$$\begin{aligned}\langle R(k)\rangle^{(0)} &= \left(\left(1-\frac{a}{2}D_1(k)\right)^{-1} D_2(k)+m\right)^{-1}\delta^4(k)\times\\ &\quad\times\left\{-2\int_q D_2(q)\mathcal{M}^{-1}(q)\left(1-\frac{a}{2}D_1(q)\right)^{-1}\right.\\ &\quad\left.+D_2(k)\int_q \mathcal{M}^{-1}(q)\left(1-\frac{a}{2}D_1(q)\right)^{-1}\right\}=0\,. \end{aligned}\tag{C.8}$$

Indeed, the first integrand is an odd function of q, while the second term is zero since $D_2(k=0)=0$.

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